FREE CONVECTION IN STRONG GRAVITY FIELDS

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Convection is considered in a narrow vertical slit with lateral temperature gradient in post-Newtonian approximation as a model problem. It is shown that the convection rate increases with gravity potential.

1. The Role of Convection in Relativistic Astrophysics. At present the investigation of motion and of the behavior of matter in strong gravity fields in the case of considerable temperature gradient with large internal energy taken into account is of great interest. Here only one aspect will be considered, namely we shall try to explain the role of free convection of a viscous heat-conductive fluid in strong gravity fields.

The role of gravity convection in astronomy has already been acknowledged for some time. Indeed, convection explains a number of astrophysical phenomena and processes; however, convective motions in objects with a high gravity potential have not as yet been investigated although their important role was pointed out by Zel'dovich and Novikov in [1]. The effect of gravity convection in strong fields is of particular interest in connection with the latest concepts and discoveries in the field of relativistic astrophysics, namely: neutron stars, quasars, superstars, residual radiation, pulsars [1, 2]. The extremal conditions mentioned previously are valid for these cosmic objects.

It follows from the Newtonian convection theory [1] that its effects are proportional to the mass of the object in whose field the convection motion is investigated. Of course, for a larger mass the convection assumes a much greater role. The role of convection also increases with the increase in temperature gradients which takes place is cosmic objects.

For objects comparable with gravity radius of the superstar of a neutron star variety convection may lead to a considerable increase in the velocity of motion; it can even exceed the parabolic velocity and result in the ejection of masses from the object. It is therefore probable that convection which causes the escape of matter may be instrumental in protecting the evolution of massive and dense objects from catastrophic collapse.

Convection in cosmic objects can lead to nonradial oscillations of considerable mass of matter which, in turn, can result in gravity radiation; the latter is of considerable interest at present in view of the latest experiments carried out by Weber [4]. Since no objects have so far been found which would cause powerful gravity radiation it is important to find out what role the contribution of the convective mechanism plays in explaining gravity radiation of cosmic objects.

Accelerated plasma ejected from the main body of a cosmic object may also generate a strong electromagnetic radiation. Here the vulcanic hypothesis of Dyson [5] is of considerable interest in accordance with which pulsar radiation takes place when plasma matter is ejected from the main layer of the neutron star because of strong convection the latter being sufficiently strong so that it pierces the outer solid crystallic layer of the neutron star.

All this points to the fact that gravity convection may have an important part to play in the evolution of cosmic objects and that a number of cosmic phenomena and effects in strong fields may be explained by convection. Therefore the effect of gravity convection is today a very important and topical study.

In the present article the general problem of motion of viscous heat-conductive fluid is first formulated within the general theory of relativity (GTR). However, since the study of such a motion would be very difficult we turn our attention to hydrodynamics in post-Newtonian approximation i.e., with an accuracy

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up to the terms (v^2/c^2) and (U/c^2), where v is velocity; U is gravity potential and c is light velocity. However, even in this case it is extremely difficult to obtain an analytic solution. Therefore, our model problem will be stationary post-Newtonian convection in a vertical slit with a lateral temperature gradient. Special characteristic features will be studied on this example of convection in strong gravity fields thus confirming its important role in astrophysical objects.

2. Viscous Heat-Conductive Fluid in GTR. It is known [3, 4] that the motion of a viscous heat-conductive fluid in the special theory of relativity (STR) is described by the relativistic hydrodynamics equations

$$T^{ik}{}_{,k} = 0, \tag{1}$$

where the comma indicates partial differentiation. The Roman subscripts and superscripts assume the values 0, 1, 2, 3, the Greek subscripts the values 1, 2, 3, and T^{ik} is the energy–momentum tensor of the viscous fluid. The expression for this tensor is given by [3]

$$T^{ik} = T_0^{ik} + \tau^{ik} = (\varepsilon + p)\,u^i u^k - p g^{ik} + \tau^{ik}, \tag{2}$$

where T_0^{ik} is the energy–momentum tensor of an ideal fluid; ε is the energy density; p is pressure; g^{ik} are metric tensor components (of signature 2); u^i are the 4-velocity components; τ^{ik} is tensor of viscous tensions [3],

$$\tau^{ik} = -\eta c\left(\frac{\partial u^i}{\partial x_k} + \frac{\partial u^k}{\partial x_i} + u^k u^e\,\frac{\partial u^i}{\partial x^e} + u^i u^e\,\frac{\partial u^k}{\partial x^e}\right) - \left(\zeta - \frac{2}{3}\eta\right) c\,\frac{\partial u_e}{\partial x^e}\,(g^{ik} + u^i u^k), \tag{3}$$

where η and ζ are the first and second viscosity coefficients respectively.

The equation of conservation of the number of particles [3]

$$(n u^i + \nu^i)_{,\,i} = 0 \tag{4}$$

should be added to Eq. (1). In the above n is the density of the number of particles, and the vector ν^i, due to heat conductivity, is written as

$$\nu^i = -\frac{\chi}{c}\left(\frac{nT}{w}\right)^2\left[\frac{\partial}{\partial x_i}\cdot\frac{\mu}{T} + u^i u^k\,\frac{\partial}{\partial x^k}\cdot\frac{\mu}{T}\right], \tag{5}$$

where χ is the heat-conduction coefficient; T is temperature; w is the heat function; μ is chemical potential. A derivation of the expressions (3) and (5) from the microscopic point of view can be found in [7].

Yet another equation, namely the state equation

$$p = p(V,\ T,\ S), \tag{6}$$

should be added to Eqs. (1) and (4). Then the system (1), (4), and (6) describes the motion of a viscous heat-conductive fluid in STR.

In the process of describing the fluid motion in GTR ordinary derivatives should be replaced by covariant ones in (1), (3), and (4); then the equations of motion in GTR are

$$T^{ik}{}_{;k} = 0, \tag{7}$$

$$(n u^i + \nu^i)_{;i} = 0, \tag{8}$$

where a semicolon indicates a covariant derivative. Einstein's gravitational equations must also be added:

$$R^{ik} - \frac{1}{2} g^{ik} R = -\varkappa T^{ik}. \tag{9}$$

The system of equations (7)-(9) and (6) describes the motion of a viscous heat-conductive fluid in GTR. In the general case it is extremely difficult to find a solution.

We shall confine our consideration to a particular but important and interesting model problem which is familar in Newtonian theory, this being the fluid motion in a narrow vertical channel in the case of lateral temperature gradient. Post-Newtonian hydrodynamics will be employed, that is, terms will be retained of the series in the powers of $1/c^2$.

3. Post-Newtonian Stationary Convection in Vertical Channel with Lateral Temperature Gradient. To solve the above problem the post-Newtonian hydrodynamics developed by Chandrasekhar [8] is employed. In such dydrodynamics terms of the series up to U/c^2 and v^2/c^2 inclusive are retained.

Some usual simplifying assumptions are now made. The case of stationary convection will be analyzed, i.e., all terms containing the derivatives with respect to time will be omitted. In the viscosity tensor (3) the second viscosity ξ is also omitted. Furthermore, squares of velocities are also omitted (which linearize the problem). Finally, one sets

$$u^i{}_{;i} = 0, \tag{10}$$

which is a relativistic analog in the usual sense of the fluid being incompressible; then the equations of post-Newtonian hydrodynamics (7) become

$$T_0^{\alpha k}{}_{;k} = \frac{\partial \tau^{\alpha\beta}}{\partial x^\beta} + \Gamma^\alpha_{00}\tau^{00} + 2\Gamma^\alpha_{\beta 0}\tau^{0\beta} + \Gamma^\alpha_{\beta\gamma}\tau^{\beta\gamma} + y_\beta \tau^{\alpha\beta} = 0. \tag{11}$$

The quantity $T_{0;k}^{\alpha k}$ was evaluated in [6]. Under our assumptions the components of the metric tensor can be written as [6]

$$g^{00} = 1 + \frac{2U}{c^2} + \frac{1}{c^4}(2U^2 + 4\Phi) + O(c^{-6}),$$
$$g^{0\alpha} = \frac{1}{c^3}4U_\alpha + O(c^{-5}), \tag{12}$$
$$g^{\alpha\beta} = -\left(1 - \frac{2U}{c^2}\right)\delta_{\alpha\beta} + O(c^{-4}),$$

and the velocity components as

$$u^0 = 1 + \frac{U}{c^2} + O(c^{-4}),$$
$$u^\alpha = \left(1 + \frac{U}{c^2}\right)\frac{v_\alpha}{c} + O(c^{-5}). \tag{13}$$

The Christoffel symbols and the quantities y_α are as follows:

$$\Gamma^0_{00} = 0, \quad \Gamma^0_{0\alpha} = -\frac{1}{c^2}\cdot\frac{\partial U}{\partial x^\alpha},$$
$$\Gamma^\alpha_{00} = -\frac{1}{c^2}\cdot\frac{\partial U}{\partial x^\alpha},$$
$$\Gamma^\alpha_{\beta 0} = -\frac{2}{c^3}\left(\frac{\partial U_\alpha}{\partial x^\beta} - \frac{\partial U_\beta}{\partial x^\alpha}\right), \tag{14}$$
$$\Gamma^\alpha_{\beta\gamma} = \frac{1}{c^2}\left(\frac{\partial U}{\partial x^\gamma}\delta_{\alpha\beta} + \frac{\partial U}{\partial x^\beta}\delta_{\alpha\gamma} - \frac{\partial U}{\partial x^\alpha}\delta_{\beta\gamma}\right),$$
$$y_\alpha = \frac{2}{c^2}\cdot\frac{\partial U}{\partial x^\alpha}.$$

In the relations (12)-(14) the gravity potential U is related to the density ρ by the Poisson equation:

$$\Delta U = -4\pi G\rho. \tag{15}$$

The potential U_α can be determined from the equation

$$\Delta U_\alpha = -4\pi G\rho v_\alpha, \tag{16}$$

and the potential Φ from the equation

$$\Delta \Phi = -4\pi G\rho\varphi, \tag{17}$$

where

$$\varphi = U + \frac{3}{2}\cdot\frac{p}{\rho}. \tag{18}$$

In view of the above the tensor of the viscous tensions can be written in the form

$$\tau^{ik} = -\eta c\left(g^{km}\nabla_m u^i + g^{im}\nabla_m u^k + u^k u^e \nabla_e u^i + u^i u^e \nabla_e u^k\right), \tag{19}$$

where the covariant derivative ∇_e is

$$\nabla_e u^i = \frac{\partial u^i}{\partial x^e} + \Gamma^i_{em} u^m. \tag{20}$$

In Eq. (11) only the terms up to $1/c^2$ should be retained. It can easily be verifield from (19) by direct computation that

$$\tau^{00} = O(c^{-1}), \quad \tau^{0\alpha} = O(c^{-1}), \tag{21}$$

and since

$$\Gamma^\alpha_{00} = O(c^{-2}), \quad \Gamma^\alpha_{\beta 0} = O(c^{-3}), \tag{22}$$

therefore the terms $\Gamma^\alpha_{00}\tau^{00}$ and $\Gamma^\alpha_{\beta 0}\tau^{0\beta}$ can be omitted since they are of a higher order of smallness than $1/c^2$. Having carried out some computations the following expression can be obtained:

$$\tau^{\alpha\beta} = \eta\left[\left(\frac{\partial v_\alpha}{\partial x^\beta} + \frac{\partial v_\beta}{\partial x^\alpha}\right) - \frac{U}{c^2}\left(\frac{\partial v_\alpha}{\partial x^\beta} + \frac{\partial v_\beta}{\partial x^\alpha}\right) + \frac{2v_\alpha}{c^2}\cdot\frac{\partial U}{\partial x^\beta} + \frac{2v_\beta}{c^2}\cdot\frac{\partial U}{\partial x^\alpha} + \frac{2v_\mu}{c^2}\cdot\frac{\partial U}{\partial x^\mu}\delta_{\alpha\beta}\right], \tag{23}$$

or, using (14) and (10) which in post-Newtonian approximation becomes

$$\frac{\partial}{\partial x^\alpha}\left[v_\alpha\left(1 + \frac{3U}{c^2}\right)\right] = 0, \tag{24}$$

one finally obtains

$$\frac{\partial \tau^{\alpha\beta}}{\partial x^\beta} + \Gamma^\alpha_{\beta\gamma}\tau^{\beta\gamma} + y_\beta \tau^{\alpha\beta} = \eta\left[\left(1 - \frac{U}{c^2}\right)\frac{\partial^2 v_\alpha}{\partial x^\beta \partial x^\beta} + \frac{5}{c^2}\cdot\frac{\partial v_\alpha}{\partial x^\beta}\cdot\frac{\partial U}{\partial x^\beta} + \frac{2}{c^2}\cdot\frac{\partial v_\gamma}{\partial x^\alpha}\cdot\frac{\partial U}{\partial x^\gamma} + \frac{v_\beta}{c^2}\cdot\frac{\partial^2 U}{\partial x^\alpha \partial x^\beta} + \frac{2}{c^2}v_\alpha\frac{\partial^2 U}{\partial x^\beta \partial x^\beta}\right]. \tag{25}$$

According to Chandrasekhar [8] one has under our assumptions

$$T^{\alpha k}_{;k} = \frac{\partial}{\partial x^\alpha}\left[\left(1 + \frac{2U}{c^2}\right)p\right] - \rho\frac{\partial U}{\partial x^\alpha} - \frac{2}{c^2}\rho\left(\varphi\frac{\partial U}{\partial x^\alpha} + \frac{\partial \Phi}{\partial x^\alpha}\right). \tag{26}$$

Finally, the stationary linearized hydrodynamic equations of viscous fluid in post-Newtonian approximation become

$$\frac{\partial}{\partial x^\alpha}\left[\left(1 + \frac{2U}{c^2}\right)p\right] - \rho\frac{\partial U}{\partial x^\alpha} - \frac{2}{c^2}\rho\left(\varphi\frac{\partial U}{\partial x^\alpha} + \frac{\partial \Phi}{\partial x^\alpha}\right) + \eta\left[\left(1 - \frac{U}{c^2}\right)\frac{\partial^2 v_\alpha}{\partial x^\beta \partial x^\beta} + \frac{5}{c^2}\cdot\frac{\partial v_\alpha}{\partial x^\beta}\cdot\frac{\partial U}{\partial x^\beta} + \frac{2}{c^2}\cdot\frac{\partial v_\gamma}{\partial x^\alpha}\cdot\frac{\partial U}{\partial x^\gamma} + \frac{1}{c^2}v_\beta\frac{\partial^2 U}{\partial x^\alpha \partial x^\beta} + \frac{2}{c^2}v_\alpha\frac{\partial^2 U}{\partial x^\beta \partial x^\beta}\right] = 0. \tag{27}$$

Standard convection equations can be obtained from Eq. (27). To this end it is assumed that

$$\rho = \rho_0 + \rho_1, \quad p = p_0 + p_1, \quad T = T_0 + T_1, \quad n = n_0 + n_1, \tag{28}$$

where the subscript zero refers to the equilibrium values of the parameters, and the subscript 1 refers to their perturbed values.

In the derivation of convection equations the equilibrium equation is, as usual, used though by now in its post-Newtonian approximation. By virtue of (27) it becomes

$$\frac{\partial}{\partial x^\alpha}\left[\left(1 + \frac{2U}{c^2}\right)p_0\right] - \rho_0\frac{\partial U}{\partial x^\alpha} - \frac{2}{c^2}\rho_0\left(\varphi\frac{\partial U}{\partial x^\alpha} + \frac{\partial \Phi}{\partial x^\alpha}\right) = 0. \tag{29}$$

Similarly as in the Newtonian case for the perturbed density value it is assumed that

$$\rho_1 = -\beta\rho_0 T_1,$$

where β is the thermal expansion coefficient. Then the hydrodynamics equations governing convection become

$$\frac{1}{\rho_0}\cdot\frac{\partial p_1}{\partial x^\alpha}\left(1+\frac{2U}{c^2}\right)+\frac{2}{c^2}\cdot\frac{p_1}{\rho_0}\cdot\frac{\partial U}{\partial x^\alpha}+\beta T_1\left[\frac{\partial U}{\partial x^\alpha}+\frac{2}{c^2}\left(\varphi\frac{\partial U}{\partial x^\alpha}+\frac{\partial\Phi}{\partial x^\alpha}\right)\right]+\frac{\eta}{\rho_0}\left[\left(1-\frac{U}{c^2}\right)\frac{\partial^2 v_\alpha}{\partial x^\beta\,\partial x^\beta}+\frac{5}{c^2}\cdot\frac{\partial v_\alpha}{\partial x^\beta}\cdot\frac{\partial U}{\partial x^\beta}+\frac{2}{c^2}\cdot\frac{\partial v_\gamma}{\partial x^\alpha}\cdot\frac{\partial U}{\partial x^\gamma}+\frac{1}{c^2}v_\beta\frac{\partial^2 U}{\partial x^\alpha\,\partial x^\beta}+\frac{2}{c^2}v_\alpha\frac{\partial^2 U}{\partial x^\beta\,\partial x^\beta}\right]=0. \quad (31)$$

If the terms of the order of $1/c^2$ are omitted the classic equation [9] is at once obtained from Eq. (31) which describes the convective motion in a narrow vertical channel in the case of lateral heating:

$$\frac{1}{\rho_0}\cdot\frac{\partial p_1}{\partial x^\alpha}+\beta T_1\frac{\partial U}{\partial x^\alpha}+\frac{\eta}{\rho_0}\cdot\frac{\partial^2 v_\alpha}{\partial x^\beta\,\partial x^\beta}=0. \quad (32)$$

The equation of conservation of particles, or otherwise the energy conservation equation is now considered.

Equation (8) can be rewritten as

$$\frac{\partial (nu^i)}{\partial x^i}+\frac{\partial v^i}{\partial x^i}+\Gamma^i_{ik}nu^k+\Gamma^i_{ik}v^k=0. \quad (33)$$

An equation for perturbed particles is now constructed from (33) bearing in mind that in the equilibrium state the relation

$$\frac{\partial v^\alpha_{(0)}}{\partial x^\alpha}+\Gamma^i_{i\alpha}v^\alpha_{(0)}=0 \quad (34)$$

is valid. It is not difficult to see that one can omit terms depending on the 4-velocity in the expression for $\nu\alpha$ since they are of a higher order of magnitude.

The fact that our heating is only lateral, i.e.,

$$T_1 = T_1(x^2), \quad (35)$$

is now taken into account and also that the problem is two-dimensional, namely that all quantities are independent of x^1; it is assumed that the gravitation force is in the opposite direction to the χ^3 axis. Then the energy equation for the perturbed quantities is

$$\frac{\partial v^2_{(1)}}{\partial x^2}+\frac{\partial (n_0u^2)}{\partial x^2}+\frac{\partial (n_0u^3)}{\partial x^3}+\Gamma^i_{i2}n_0u^2+\Gamma^i_{i3}n_0u^3=0. \quad (36)$$

By using (14) it is not difficult to show that

$$\Gamma^i_{i2}=\frac{2}{c^2}\cdot\frac{\partial U}{\partial x^2}, \quad (37)$$

$$\Gamma^i_{i3}=\frac{2}{c^2}\cdot\frac{\partial U}{\partial x^3}. \quad (38)$$

Since it was assumed that the slit is narrow the gradient of the potential (37) can be omitted.

Here by the perturbed quantity $\nu^2_{(1)}$ one understands the expression

$$v^2_{(1)}=\frac{\chi}{c}\cdot\frac{n_0}{w_0}\cdot\frac{\partial T_1}{\partial x^2}. \quad (39)$$

In the derivation of (39) the thermodynamic identity

$$d\,\frac{\mu}{T} = -\frac{w}{nT^2}\,dT + \frac{1}{nT}\,dp \tag{40}$$

was employed. In view of what was said above the energy conservation equation (36) becomes

$$\frac{\partial v^2_{(1)}}{\partial x^2} + \frac{\partial (n_0 u^2)}{\partial x^2} + \frac{\partial (n_0 u^3)}{\partial x^3} + \Gamma^i_{i3} n_0 u^3 = 0, \tag{41}$$

or, by using (13),(38), and (39) it becomes

$$\frac{\partial^2 T_1}{(\partial x^2)^2} + \frac{\partial}{\partial x^\alpha}\left[v_\alpha\left(1 + \frac{3U}{c^2}\right)\right] = 0. \tag{42}$$

By using (24) which is valid the equation of energy conservation can finally be written as follows:

$$\frac{\partial^2 T_1}{(\partial x^2)^2} = 0. \tag{43}$$

By direct substitution of the classical equation governing the convection in a narrow vertical slit with lateral heating,

$$v_2 = 0, \qquad v_3 = v_3(x^2), \tag{44}$$

into (13) and (24) one finds that the solution (44) does not satisfy Eqs. (31) and (24). This indicates that in post-Newtonian approximation the profile of convection velocities will no longer be greater than cubic parabola.

The specific feature of GTR in the systems (24) and (31) lies also in that in addition to a more complicated dependence on the potential gradient there occurs a dependence on the potential itself (potential difference).

It is extremely difficult to solve systems (24),(31), and (43) in the general case which should obviously be done numerically. However, an exact analytic solution for these systems can be obtained in the particular and important case.

Let us consider convection in an object, for example, in a neutron star. The standard parameters of such a star are: radius R = 10 km, mass, approximately solar $\sim 2 \cdot 10^{33}$ g. If convection is now considered in a slit of length r = 0.1 at the distance R to the center, then one can assume that the quantity $U/c^2 \approx 0.15$ is approximately constant and that the quantity $(1/c^2) \cdot (\partial U/\partial R)$ can be everywhere omitted in the Eqs. (31) and (24). In this case Eqs. (24) and (31) assume a quasiclassical form:

$$\frac{\partial v_2}{\partial x^2} + \frac{\partial v_3}{\partial x^3} = 0, \tag{45}$$

$$\frac{\eta}{\rho} \cdot \frac{\partial^2 v_3}{(\partial x^2)^2} + \left(1 + \frac{U}{c^2}\right)\beta T_1 \frac{\partial U}{\partial x^3} = 0. \tag{46}$$

It is obvious that a solution of the systems (45) and (46) can be sought, similarly as in the classical case, in the form (44). Then it follows directly from Eq. (46) that convection is uprated by a quantity proportional to U/c^2.

The obtained result confirms the hypothesis of Dyson [5] of the important part played by convection in neutron stars in which pulsar radiation may be explained by such a mechanism. The obtained results can also be applied to superstars of Hoyle and Fowler[10], where the quantity U/c^2 is approximately equal to 0.2-0.3.

NOTATION

v_α is the 3-velocity component;
U is the gravity potential;
c is the velocity of light;
u^i is the 4-velocity component;

η, ζ are the first and second viscosity coefficients respectively;
χ is the heat conductivity coefficient;
T is the temperature;
w is the heat function;
μ is the chemical potential;
ρ is the density;
p is the pressure;
ε is the energy density;
T^{ik} are the energy–momentum tensor;
Γ^{i}_{jk} are the Christoffel symbol;
β is the heat expansion coefficient.

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